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A GENERAL TYPE OF EXPANSIVE DIFFEOMORPHISMS

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Abstract. In the paper, we consider a general type of measure expansive diffeomorphisms which is called expanding measure. Using this notion, we prove that if a transitive set Λ is robustly expanding measure then Λ is hyperbolic.

1. Introduction

Let M be a closed smooth manifold without boundry with dim$M \geq 2$. Expansivity was introduced in [16]. A dffeomorphism $f : M \to M$ is -*expansive* if there exists a constant $\delta > 0$ (called a expansive constant) such that if for any $x, y \in M$, $d(f^i(x), f^i(y)) < \delta$ for all $i \in \mathbb{Z}$, then $x = y$. General notion of expansiveness such as N-expansive([12]), measure-expansive([13]), continuum-wise expansive([4]), and expanding measure([2]) are closely related to hyperbolic structure. For example, Mañé proved in [11] that if a diffeomorphism f belongs to the set of expansive diffeomorphisms of M, then it is qausi-Anosov. Here, a diffeomorphism f is *quasi-Anosov* if for all $v \in TM \setminus \{0\}$, the set $\{\|Df^n(v)\| : n \in \mathbb{Z}\}$ is unbounded. Also, if a diffeomorphism f belongs

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to the set of N-expansive([5]), measure expansive([15]), and continuum-wise expansive([14]) then it is quasi-Anosov. From Artigue and [1] and [12], we see that expansiveness $\Rightarrow$ N-expansiveness $\Rightarrow$ measure expansiveness $\Rightarrow$ continuum-wise expansiveness.

By the previous result, we consider a closed f-invariant set $\Lambda \subset M$. We say that a closed f-invariant set $\Lambda \subset M$ is *robustly* $\mathcal{P}$ if there exists a C^1 neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that $\Lambda = \bigcap_{n\in\mathbb{Z}} f^n(U)$ and for any $g \in \mathcal{U}$, $U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is $\mathcal{P}$, where U_g is the continuation of Λ.

In the definition, $\mathcal{P}$ is replaced by expansive, N-expansive, measure expansive, continuum-wise expansive, and expanding measure.

A compact f-invariant set $\Lambda \subset M$ is said to be *hyperbolic* for f if there is a continuous Df-invariant splitting $T_\Lambda M = E^s \oplus E^u$ with two constants $C > 0$ and $0 < \lambda < 1$ such that (i) $\|Df^n|_{E^s(x)}\| \leq C\lambda^n$ for any $x \in \Lambda$ and $n \geq 0$, and (ii)$\|Df^{-n}|_{E^u(x)}\| \leq C\lambda^n$ for any $x \in \Lambda$ and $n \geq 0$. Lee and Park [10] proved that if a transitive set Λ is robustly expansive then Λ is hyperbolic, Lee [6] proved that a transitive set Λ is robustly continuum-wise expansive then Λ is hyperbolic. About these results, Lee considered weak measure expansiveness and asymptotic measure expansiveness (see [8, 9]). In the paper, we will prove the following.

Theorem A Let $\Lambda \subset M$ be a transitive set of f. If Λ is robustly expanding measure for f then Λ is hyperbolic for f.

2. Proof of Theorem A

Let M be as before, and let $f : M \to M$ be a diffeomorphism. Denote by $\mathcal{M}(M)$ the set of all Borel probability measures on M with the weak* topology and $\mathcal{M}_f(M)$ the set of all Borel probability invariant measure on M. It is known that $\mathcal{M}_f(M) \subset \mathcal{M}(M)$.

DEFINITION 2.1. For any $\mu \in \mathcal{M}_f(M)$, a closed f-invariant set Λ is said to be an *expanding measure* if there is a constant $\delta > 0$ (called an expanding measure constant) such that

$$\mu(\Gamma_\delta(x, f) \setminus Orb_f(x)) = 0$$

for all $x \in M$, where $Orb_f(x)$ is the orbit of x and $\Gamma_\delta(x, f) = \{y \in M : d(f^i(x), f^i(y)) \leq \delta \ \forall i \in \mathbb{Z}\}$.

LEMMA 2.2. *Let $I \subset M$ be a closed arc. If $f|_I : I \to I$ is the identity then I is not expanding measure.*

Proof. Suppose I is an expanding measure. Let ν be the normalized Lebesgue measure on I. Then we have an invariant measure $\mu \in \mathcal{M}(M)$ such that

$$\mu(C) = \frac{\nu(I \cap C)}{\nu(I)} = \nu(I \cap C),$$

for any Borel set $C \subset M$. It is clear that μ is a non-atomic measure. Since $f|_I : I \to I$ is the identity map, $Orb(x) = \{x\}$ for all $x \in I$. Then we see that $I \setminus Orb(x) = I - \{x\} = I_1 \cup I_2$ and $I_1 \cap I_2 = \emptyset$. Also, we see that $\mu(I \setminus Orb(x)) = \mu(I_1) + \mu(I_2) \neq 0$. This is a contradiction. □

The following lemma is called franks' lemma [3] which is a useful notion for the C^1 perturbations.

LEMMA 2.3. *Let $\mathcal{U}(f)$ be a C^1 neighborhood of a diffeomorphism $f : M \to M$. Then there exist a $\epsilon > 0$ and a C^1 neighborhood $\mathcal{U}_0(f) \subset \mathcal{U}(f)$ of f such that if $g \in \mathcal{U}_0(f)$, a finite set $A = \{x_1, x_2, \cdots, x_N\}$, a neighborhood W of A and $L_i (i = 1, \ldots, N)$ are linear maps $L_i : T_{x_i}M \to T_{g(x_i)}M$ satisfying $\|L_i - D_{x_i}g\| \leq \epsilon$ for all $1 \leq i \leq N$, there exists $\widehat{g} \in \mathcal{U}(f)$ satisfying $\widehat{g}(x) = g(x)$ if $x \in \{x_1, x_2, \cdots, x_N\} \cup (M \setminus W)$ and $D_{x_i}\widehat{g} = L_i$ for all $1 \leq i \leq N$.*

LEMMA 2.4. *Let $\Lambda \subset M$ be a closed set. If Λ is robustly expanding measure, then every periodic point $p \in \Lambda$ is hyperbolic.*

Proof. Suppose by contradiction there is a periodic point $p \in \Lambda$ which is not hyperbolic. Then $D_p f^{\pi(p)}$ has an eigenvalue λ with $|\lambda| = 1$. For simplicity, we assume that p is a fixed point. As Lemma 2.3, there is a diffeomorphism g C^1 close to f such that g has a closed small arc $\mathcal{I}_p$ if $\lambda \in \mathbb{R}$ or a closed small disk C_p if $\lambda \in \mathbb{C}$(see more details [7]). For the closed sets, we first consider $\mathcal{I}_p$ which has the following properties; $g(\mathcal{I}_p) = \mathcal{I}_p$, and $g|_{\mathcal{I}_p} : \mathcal{I}_p \to \mathcal{I}_p$ is the identity map. As Lemma 2.2, $\mathcal{I}_p$ is not expanding measure for g. This is a contradiction. Now we consider C_p. Then we have the following properties: $g^k(C_p) = C_p$ for some $k \in \mathbb{Z}$, and $g^k|_{C_p} : C_p \to C_p$ is the identity map.

Let m be the normalized Lebesgue measure on C_p. We define an invariant measure $\mu \in \mathcal{M}(M)$ by

$$\mu(C) = \frac{1}{k}\sum_{i=0}^{k-1} m(g^{-i}(g^{i}(C_p) \cap C)),$$

for any Borel set $C \subset M$. Also, it is a non-atomic measure. Since $g^k|_{C_p} : C_p \to C_p$ is the identity map, we have that

$$\mu(C_p \setminus Orb(x)) = \frac{1}{k}\sum_{i=0}^{k-1} m(g^{-i}((C_p \setminus Orb(x)) \cap g^{i}(C_p)).$$

Since $m(C_p \setminus Orb(x)) \neq 0$, we have $\mu(C_p \setminus Orb(x)) \neq 0$. This is a contradiction. This means that if a transitive set Λ is robustly expanding measure then every periodic point in Λ is hyperbolic. □

A closed f-invariant set $\Lambda \subset M$ satisfies a *local star condition* if there are a C^1 neighborhood $\mathcal{U}$ of f and a neighborhood U of Λ such that, for any $g \in \mathcal{U}$, every periodic point $p \in U_g = \bigcap_{n\in\mathbb{Z}} g^n(U)$ is hyperbolic.

LEMMA 2.5. *Let $\Lambda \subset M$ be a closed set. If Λ is robustly expanding measure then Λ satisfies a local star condition.*

Proof. Since Λ is robustly expanding measure, by Lemma 2.4, every periodic points in Λ is hyperbolic. This means that Λ satisfies a local star condition. □

LEMMA 2.6. [7] *Let $\Lambda \subset M$ be a transitive set. If Λ satisfies a local star condition then Λ is hyperbolic.*

Proof of Theorem A Since Λ is robustly expanding measure, by Lemma 2.5, Λ satisfies a local star condition. As Lemma 2.6, Λ is hyperbolic. □

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